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ON THE FORMATION OF THE DISTURBANCE-FIELD STRUCTURE IN A TRANSITIONAL BOUNDARY LAYER

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Introduction. The nonlinear evolution of disturbances in boundary-layer flows is a determining factor of the laminar–turbulent transition (LTT). Weakly nonlinear theory has been developed [1, 2] to study the evolution of weak pulsations. This theory is based on the assumption of local proximity of the hydrodynamic field to the distribution formed by linear disturbances. Viscous effects dominate in this process. Nonlinearity introduces corrections of higher order (with respect to fluctuation amplitude), which can, nevertheless, change considerably the spectrum and amplification rates of these fluctuations. Weakly nonlinear theory ensures a successful interpretation of subharmonic S-transition phenomena. The latter occurs at low initial disturbance amplitudes. It is characterized by an outstripping growth of low-frequency, three-dimensional, Tollmien–Schlichting waves, in particular, subharmonic waves with respect to the wave revealed in the initial stage. The main mechanism involves nonlinear resonance interactions in wave triads [2–5].

The region of applicability of weakly nonlinear theory is rather limited. As the disturbance intensity increases, nonlinearity can play a dominant role in flow-field structuring. This occurs primarily in the critical layer (CL) of the wave, in which the phase velocity of the wave coincides with the local flow velocity. Along with the wall layer (WL), the CL region is of significance for the mechanism of energy exchange between the disturbances and the mean flow [6–10].

Three CL types are identified in accord with which effect is most pronounced: unsteadiness, viscosity, or nonlinearity. Their thicknesses are [10]

$$l_t = \gamma/\alpha, \qquad l_\nu = (\alpha\,\mathrm{Re})^{-1/3}, \qquad l_N = A^{1/2},$$

where α, A, and γ are the typical wavenumber, amplitude, and increment of the wave, and Re is the Reynolds number. The region of applicability of nonlinear theory corresponds to the condition $l_N \ll (l_t, l_\nu)$ [10]. A change of the CL type transforms the pulsation field structure and the mean flow. The turbulization pattern becomes different.

The evolution of quasi-periodic disturbances with various CL types has been successfully investigated in free shear layers [11–13], where the CL behavior almost completely determines the wave energetics. The study of near-wall flows, for which the wall influence is essential, is more problematic. Solutions for neutral wave disturbances were constructed in [6, 7, 14]. Some aspects of variation of flow parameters in the CL vicinity were considered in [9]. A more detailed procedure for obtaining evolution equations was suggested in [8, 10]. The analysis in these papers, however, was based on a number of heuristic assumptions of a correlation between the amplitude growth rate and the wave shape, and was restricted to the case of a mono-harmonic disturbance whose CL is asymptotically far from a rigid boundary.

The typical features of disturbances with nonlinear CLs revealed in [6–14] are intense generation of high-frequency harmonics, abrupt transformation of the vorticity pulsation field, and deformation of the mean profile in the CL vicinity. Apparently, these features affect the interaction efficiency of fluctuations of different wavelengths. In particular, they lead to parametric excitation of low-frequency background pulsations which play a determining role in the S-transition. This can lead to qualitative changes in the LTT scenario during the formation of a nonlinear CL regime.

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In the present paper, we propose a method for analysis of the evolution of wave disturbances in plane-parallel flows of the boundary-layer type. This method is equally suitable in regions of linear, weakly nonlinear, and highly nonlinear flow evolution (free of restrictions on the type and position of critical layers of waves). Asymptotic averaging and matched asymptotic expansions were used for constructing solutions. The relative efficiency of parametric excitation of background subharmonic pulsations by a primary wave with linear and nonlinear CLs was studied. A decrease in the rate of parametric growth of subharmonics in the field of a strong wave was found. Threshold amplitudes for the formation of various types of transition in the boundary layer were estimated. It is concluded that the S- and K-transitions are related to regimes of weakly nonlinear evolution and nonlinear CL of the primary wave.

Method of Solving the Evolution Problem. Let us consider a plane-parallel flow of the boundary-layer type. The flow field is described by a nondimensional stream function $\tilde{\psi}$ that obeys the equation

$$\frac{\partial}{\partial t}\Delta\tilde{\psi} + \tilde{u}\frac{\partial}{\partial x}\Delta\tilde{\psi} + \tilde{v}\frac{\partial}{\partial y}\Delta\tilde{\psi} - \frac{\Delta^2}{\mathrm{Re}}\tilde{\psi} = 0, \tag{1}$$

where $\tilde{u} = \partial\tilde{\psi}/\partial y$, $\tilde{v} = -\partial\tilde{\psi}/\partial x$, $\tilde{\psi} = \Psi(y) + \varepsilon\psi(x,y,t)$, $\varepsilon \ll 1$, and Ψ and ψ refer to the undisturbed flow and the disturbance, respectively.

Let us examine the evolution of quasi-harmonic disturbances

$$\psi = \sum_s \psi_s(y,t)\exp(i\theta_s), \qquad \theta_s = \alpha_s x - \omega_s t, \qquad \left|\frac{1}{\omega_s\psi_s}\frac{\partial}{\partial t}\psi_s\right| \sim \mu \ll 1. \tag{2}$$

Substituting (2) into (1) and averaging over fast phases [2, 5], we obtain

$$L\psi_s \equiv \left\{\Delta\psi_s - \frac{U''}{U-c_s}\psi_s\right\} = \frac{i}{\alpha(U-c_s)}\left\{\frac{\partial}{\partial t}\Delta\psi_s - \frac{\Delta^2}{\mathrm{Re}}\psi_s - \varepsilon J_{ik}h^s_{i+k}\right\} \equiv Q_s,$$
$$\psi_s = \frac{\partial}{\partial y}\psi_s = 0 \quad (y=0), \qquad \Gamma\psi_s \equiv \left\{\frac{\partial}{\partial y}\psi_s + \alpha_s\psi_s\right\} = 0 \quad (y = y_\infty), \quad \psi_{s0} = \psi_s \quad (t=0), \tag{3}$$

where $U = d\Psi/dy$; the prime denotes a derivative with respect to y. In this case,

$$c_s = \omega_s/\alpha_s, \quad \Delta\psi_s = \left(\frac{\partial^2}{\partial y^2} - \alpha_s^2\right)\psi_s,$$

$$J_{ik} = \left\{\frac{\partial}{\partial x}\psi_i\frac{\partial\Delta}{\partial y}\psi_k - \frac{\partial}{\partial y}\psi_k\frac{\partial\Delta}{\partial x}\psi_i + \frac{\partial}{\partial x}\psi_k\frac{\partial\Delta}{\partial y}\psi_i - \frac{\partial}{\partial y}\psi_i\frac{\partial\Delta}{\partial x}\psi_k\right\},$$

$$h^s_{i+k} = \frac{1}{2\pi}\int_0^{2\pi}\exp i(\theta_s - \theta_i - \theta_k)\,d\theta.$$

With accuracy to $O(\mu, \varepsilon, \mathrm{Re}^{-1})$, we obtain the Rayleigh problem

$$L\psi_s \equiv 0, \qquad \psi_s = 0 \quad (y=0), \qquad \Gamma\psi_s = 0, \qquad \psi_{s0} = \psi_s(y,0), \tag{4}$$

whose solution is of the form

$$\psi_s = A_j\varphi_{1j}(y)\delta_{sj}, \qquad \omega_j = \omega(\alpha_j), \qquad \delta_{sj} = \begin{cases} 1, & s=j, \\ 0, & s\neq j, \end{cases}$$

where $\varphi_{1j} = \varphi_1(y,\alpha_j)$ and $\omega(\alpha_j)$ are the eigenfunctions and eigenvalues that correspond to the wavenumber α_j. It is assumed that the initial disturbance ψ_{s0} is represented as a superposition of Rayleigh solutions, which below are distinguished by the subscripts i, j, and k. The subscripts s, r, and p correspond to waves whose phases are pulsations ($\theta_s = \pm\theta_i \pm \theta_k \pm \cdots$), and, hence, ω_s and α_s are not necessarily connected through the dispersion relation of the linear problem.

Introducing the normalization $\varepsilon_1 = \varepsilon/\mu$ and $\nu = 1/\mathrm{Re}\,\mu$ without loss of generality, we shall seek a solution of (3) in the form

$$\psi_s = \sum_{m=0} \mu^m \psi_s^{(m)}, \qquad \frac{d}{dt}A_j = \sum_{m=1} \mu^m P_j^{(m)}, \tag{5}$$

where $\psi_s^{(m)}$ and $P_j^{(m)}$ are functions and operators that take into account the distortion of primary Rayleigh waves (4). These distortions are caused by the unsteady, nonlinear, and viscous effects neglected in (4). It is convenient to distinguish the contribution $\tilde{f}_s$ made by singular regions to the stream function. These regions are CLs in which $(U(y) - c_s) \to 0$ as $y \to y_s$, and WLs, which ensure $\partial\psi_s/\partial y = 0$ $(y = 0)$. Within these regions, expansion (5) is no longer regular.

Outside of these layers,

$$\psi_s^{(m)} = f_s^{(m)} + \tilde{f}_s^{(m)}, \quad \tilde{f}_s^{(m)} = \Phi_s^{(m)} + w_s^{(m)}, \quad f_s^{(0)} = \psi_s^{(0)} = A_j\varphi_{1j}\delta_{sj}, \quad \tilde{f}_s^{(0)} = 0. \tag{6}$$

Here $\tilde{f}_s^{(m)}$ is a singular part of the solution whose components $[\Phi_s^{(m)}$ and $w_s^{(m)}]$ are due to the CL and WL, respectively, and $f_s^{(m)}$ is a regular part of the solution, for which the system

$$\begin{gathered} Lf_s^{(m)} = \frac{i}{\alpha_s(U - c_s)}\Big\{P_j^{(m)}\Delta\varphi_{1j}\delta_{js} + \frac{\partial\Delta}{\mu\partial t}\psi_s^{(m-1)} - \nu\Delta^2\psi_s^{(m-1)} - \varepsilon_1 J_s^{(m-1)}\Big\} \equiv Q_s^{(m)}, \\ f_s^{(m)} + \tilde{f}_s^{(m)} = 0 \quad (y = 0), \qquad \Gamma(f_s^{(m)} + \tilde{f}_s^{(m)}) = 0, \qquad J_s^{(m)} = \sum_{n=0} J(\psi_i^{(m-n)}, \psi_k^{(n)})h_{i+k}^s \end{gathered} \tag{7}$$

is valid. This is equivalent to $L\psi_s^{(m)} = Q_s^{(m)} + L\tilde{f}_s^{(m)}$, $\psi_s^{(m)}(t,0) = 0$, and $\Gamma\psi_s^{(m)} = 0$.

The no-slip conditions take the form

$$\frac{\partial}{\partial y}(f_s^{(m)} + \Phi_s^{(m)}) + \frac{\partial}{\partial y}\mu w^{(m+1)} = 0. \tag{8}$$

Note that the boundary conditions for $y = 0$ in the form of (7) and (8) are valid if the CL does not lie on the wall $(y_s \gg y = 0)$. Consequently, representation (6) holds. Otherwise, the conditions $\psi_s = \partial\psi_s/\partial y = 0$ $(y = 0)$ should be used directly.

We now solve problem (5)–(8). For long-wave disturbances $(\alpha_s \ll 1)$, the system of fundamental solutions of the Rayleigh equations can be represented by the Heisenberg functions $\varphi_{1s}(y)$ and $\varphi_{2s}(y)$:

$$\begin{gathered} \varphi_{1s} = \varphi_1(y, \alpha_s) = (U - c_s)\int_0^y \frac{dy}{(U - c_s)^2} + O(\alpha_s^2) \simeq -\frac{U - c_s}{U_s'^2}\Big\{\frac{1}{z_s} + \frac{1}{y_s} + \frac{U_s''}{U_s'}\ln\Big|\frac{z_s}{y_s}\Big| - g_s\Big\}, \\ \varphi_{2s} = \varphi_2(y, \alpha_s) = (U - c_s) + O(\alpha_s^2). \end{gathered} \tag{9}$$

Here

$$g_s = \int_{-y_s}^{z_s} \Big\{\frac{U_s'^2}{(U - c_s)^2} - \frac{1}{z_s^2} + \frac{U_s''}{U_s'}\frac{1}{z_s}\Big\}dz_s; \qquad U_s = U(y_s) = c_s; \qquad z_s = y - y_s.$$

From (7) we obtain (outside the CL regions)

$$\psi_s^{(m)} = \Big\{\varphi_{1s}(y)\int_0^y \varphi_{2s}(Q_s^{(m)} + L\tilde{f}_s^{(m)})\,dy - \varphi_{2s}(y)\int_0^y \varphi_{1s}(Q_s^{(m)} + L\tilde{f}_s^{(m)})\,dy\Big\}V_s^{-1}, \tag{10}$$

where $V_s = \varphi_{1s}d\varphi_{2s}/dy - \varphi_{2s}d\varphi_{1s}/dy$ is the Wronskian of the system.

For $m = 0$, Eqs. (7) reduce to (4). The condition $\varphi_{1s} = 0$ $(y = 0)$ is fulfilled automatically, and the boundary condition on y_∞ for $s = j$,

$$\Gamma\varphi_{1j} = \frac{1}{U_0 - c_j} - \alpha_j\frac{U_0 - c_j}{U_j'^2}\Big(\frac{1}{z_j} + \frac{1}{y_j} + \frac{U_j''}{U_j'}\ln\Big|\frac{z_j}{y_j}\Big| - g_j\Big) = 0, \tag{11}$$

$[U_0 = U(y_\infty)]$ establishes the dispersion relationship $\omega_j = \omega(\alpha_j)$.

For $s = j$ from (10) with allowance for $\varphi_{1j}(0) = \Gamma\varphi_{1j} = 0$, we obtain solvability conditions for system (7):

$$\int_0^{y_\infty} \varphi_{1j}(Q_j^{(m)} + L\tilde{f}_j^{(m)})\,dy = \int_0^{y_\infty} \varphi_{1j}Q_j^{(m)}\,dy + \tilde{f}_j^{(m)}(t, y_\infty) = 0.$$
$$\int_0^{y_\infty} \varphi_{1j}(Q_j^{(m)} + L\tilde{f}_j^{(m)})\,dy = \int_0^{y_\infty} \varphi_{1j}Q_j^{(m)}\,dy + \tilde{f}_j^{(m)}(t, 0). \tag{12}$$

They establish a correlation between the operators $P_j^{(m)}$ and the functional values of the functions $\psi_s^{(m-n)}$ determined in previous orders. Then, (5) leads to an evolution equation for $A_j(t)$. (Note that the second equation of (12) is valid only for $y_s \gg 0$.)

As noted above, $\tilde{f}_s^{(m)}$ — a singular part of the solution — is due to the CL and WL.

Isolated CLs arise in the vicinity of $y \sim y_s$ due to singular terms on the right-hand side of (7) of the form

$$Q_s \sim a_{rs}(U - c_r)^{-n} + a_{ps}(U - c_p)^{-m} + \dots$$

$[a_{rs}(y, t)$ are regular functions$]$ and lead to the corrections

$$\Phi_s^{(m)} = \sum_r (B_{rs}^{(m)}(t)\varphi_{1s} + D_{rs}^{(m)}(t)\varphi_{2s}). \tag{13}$$

Here $B_{rs}^{(m)}$ and $D_{rs}^{(m)}$ can take various values above and below the CL region ($y \simeq y_r$):

$$\Delta B_{rs}^{(m)} = B_{rs}^{(m)}|_{y_r+0} - B_{rs}^{(m)}|_{y_r-0} \neq 0, \quad \Delta D_{rs}^{(m)} = D_{rs}^{(m)}|_{y_r+0} - D_{rs}^{(m)}|_{y_r-0} \neq 0.$$

The WL influence function $w_s^{(m)}$ corresponds to the decreasing (with derivatives) solution $w_s^{(m)} \to 0$ ($y \to y_\infty$). With allowance for that, (12) takes the form

$$\int_0^{y_\infty} \varphi_{1j}Q_j^{(m)}\,dy + \sum_r D_{rj}^{(m)}\Big|_{y_\infty} = 0, \quad \int_0^{y_\infty} \varphi_{1j}Q_j^{(m)}\,dy + \Big(\sum_r D_{rj}^{(m)} + w_j^{(m)}\Big)\Big|_{y=0} = 0, \tag{14}$$

or, what is the same,

$$\fint_0^{y_\infty} \varphi_{1j}Q_j^{(m)}dy = w_j^{(m)}(t, 0) - \sum_r \Delta D_{rj}^{(m)},$$

where the integral is taken in the sense of the basic value (except for regions containing singularities).

It is obvious from (5) and (14) that dA_j/dt, along with the terms $O(\varepsilon, \mathrm{Re}^{-1}, w_j^{(m)}(0, t))$, is determined by the magnitude of the jump $\Delta D_{rj}^{(m)} \approx D\Delta y_r$, which is of the order of the CL thickness Δy_r. A parameter μ can then be introduced as a characteristic of the LC thickness: $\mu \approx \max_r \Delta y_r$.

We now determine w_s. We introduce formally internal variables $\eta = \eta_j = z_j/\mu$. Equation (1) for the stream function $\tilde{\psi}(x, \eta, \tau)$ takes the form

$$\Big\{\frac{\gamma}{\mu}\frac{\partial}{\partial\tau} + \mu^{-2}\Big(\frac{\partial\tilde{\psi}}{\partial\eta}\frac{\partial}{\partial x} - \frac{\partial\tilde{\psi}}{\partial x}\frac{\partial}{\partial\eta}\Big) - \lambda\frac{\partial^2}{\partial\eta^2}\Big\}\Omega = \mu^2\Big\{\mu^{-2}\Big(\frac{\partial\tilde{\psi}}{\partial x}\frac{\partial^3}{\partial x^2\partial\eta} - \frac{\partial\tilde{\psi}}{\partial\eta}\frac{\partial^3}{\partial x^3}\Big)\tilde{\psi} + 2\lambda\frac{\partial^2}{\partial x^2}\Omega + \mu^2\lambda\frac{\partial^2}{\partial\eta^2}\Omega\Big\}. \tag{15}$$

Here $\lambda = (\mathrm{Re}\,\mu^3)^{-1}$, $\tau = \gamma t$, $\Omega = \partial^2\tilde{\psi}/\partial\eta^2$, and $\gamma = |\partial \ln\Omega/\partial t|$ is the unsteadiness parameter.

According to (6), outside the CL,

$$\tilde{\psi} = \Psi + \varepsilon\sum_k A_k\varphi_{1k}\exp(i\theta_k) + \varepsilon\mu(f^{(1)} + \Phi^{(1)} + w^{(1)}) + O(\varepsilon^2, \varepsilon\mu^2, \varepsilon\nu) \tag{16}$$

and $w^{(1)}$ is found from the equation

$$\Big\{\frac{\gamma}{\mu}\frac{\partial}{\partial\tau}+\frac{U-c_j}{\mu^2}\frac{\partial}{\partial x_1}+2\frac{\varepsilon}{\mu^2}\frac{\partial}{\partial x_1}\sum_k(\mathrm{Im}\,A_k\varphi_{1k}e^{i\Delta\theta_k})\frac{\partial}{\partial\eta}-\lambda\frac{\partial^2}{\partial\eta^2}\Big\}\Omega_w^{(1)}=0, \qquad (17)$$

where $x_1=x-c_jt$, $\Delta\theta_k=\Delta\alpha_kx-\Delta c_kt$, $\Delta(\alpha_k,c_k)=(\alpha_k,c_k)-(\alpha_j,c_j)$, and

$$\frac{\partial^2}{\partial\eta^2}w^{(1)}=\Omega_w^{(1)}, \qquad \Big(\Omega_w^{(1)},\ \frac{\partial}{\partial\eta}\Omega_w^{(1)}\Big)\to 0 \text{ for } \eta\to\frac{|y_\infty-y_j|}{\mu}.$$

For $y_k\gg\mu$, the wall condition is

$$w^{(1)}=-(f^{(1)}+\Phi^{(1)}), \quad \frac{\partial}{\partial\eta}w^{(1)}=-\frac{\partial f^{(0)}}{\partial\eta}\mu^{-1}.$$

The subsequent orders $\sim\mu^n$ are determined from inhomogeneous equations wherein the basic operator coincides with (17).

To determine the coefficients $(B,D)_{rs}^{(m)}$ and $\Phi_s^{(m)}$, one should solve the problem in internal CL regions. The standard procedure of matched asymptotic expansion is used [7, 12]. Equations for $\psi_{iN}=\tilde\psi(\tau,\eta,x)$ within the CL ($|y-y_k|<\Delta y_k$) coincide in form with (15); the boundary condition is the expansion of $\tilde\psi$ (16) for $|y-y_k|\to 0$ in the variables $\eta_k=z_k/\mu$ [$\tilde\psi\to\psi_{iN}(|\eta_k|\to\infty)$].

For $y_s\sim\mu$ (the CL is on the wall), the conditions at $y=0$ ($\eta_s=-y_s/\mu$) do not allow the separation into "responses" (6), and, instead of matching with $\tilde\psi$ as $\eta\to-\infty$, they should be written in the general form $\psi_{iN}=\partial\psi_{iN}/\partial\eta=0$ $(y=0)$.

The iterative procedure considered above has a self-consistent character and allows one to describe the evolution of two-dimensional wave disturbances with an arbitrarily specified accuracy by solving the system (7), (12), (15), and (17). The CL type and its position relative to the wall do not impose restrictions on the method. The procedure is readily extended to the case of spatial–temporal evolution of three-dimensional waves.

Evolution of Mono-Harmonic Disturbance. To illustrate the above method and make a comparison with the previous results, we study in more detail the evolution of an isolated wave.

In this case, $j=1$, and from (2) and (5), taking into account the uniqueness of the CL, we obtain

$$\psi=\sum_{s=0}\sum_{m=0}\mu^m\psi_s^{(m)}\exp(i\theta_s), \qquad \frac{dA}{dt}=\sum_{m=1}\mu^mP^{(m)},$$
$$\psi_s^{(m)}=f_s^{(m)}+\tilde f_s^{(m)}, \quad \alpha_s=s\alpha, \quad c_s=c, \quad s=0,\,1,\,2,\,\ldots\,. \qquad (18)$$

We consider the case $\mu\ll y_1$ (the CL is asymptotically far from the wall). From (9) follows the recurrent systems

$$Lf_1^{(0)}=0, \quad \Gamma f_1^{(0)}=0, \quad f_1^{(0)}(t,0)=0, \quad \frac{\partial}{\partial y}f_1^{(0)}(t,0)\neq 0 \qquad (19)$$

for $m=0$ and

$$Lf_1^{(1)}=Q_1^{(1)}\equiv\frac{iU''\varphi_{11}P^{(1)}}{\alpha(U-c)^2},\ \Gamma\psi_1^{(1)}=0,\ \psi_1^{(1)}(t,0)=0,\ \frac{\partial}{\partial y}f_1^{(0)}+\frac{\partial}{\partial y}\mu w_1^{(1)}=0\ (y=0); \qquad (20)$$

$$Lf_2^{(1)}=Q_2^{(1)}=i\varepsilon_1\frac{J_2^{(1)}}{2\alpha(U-c)},\ \Gamma f_2^{(1)}=0,\ f_2^{(1)}(t,0)=0 \qquad (21)$$

for $m=1$.

The correction to the mean flow $f_0^{(1)}$ that arises in this order of expansion is determined only in the next approximation (since $J_0^{(1)}=\alpha_0\equiv 0$).

For $m = 2$, we have

$$Lf_1^{(2)} = \frac{i}{\alpha(U-c)}\Big\{P^{(2)}\frac{U''\varphi_{11}}{U-c} + \frac{\partial\Delta}{\mu\partial t}\psi_1^{(1)} - \varepsilon_1 J_1^{(2)} - \Delta\frac{\nu A\varphi_{11}U''}{U-c}\Big\},$$

$$\Gamma\psi_1^{(2)} = 0,\ \psi_1^{(2)} = \frac{\partial}{\partial y}(f_1^{(1)} + \Phi_1^{(1)} + \mu w_1^{(2)}) = 0 \qquad (y = 0), \tag{22}$$

$$J_1^{(2)} = J_{10}(f_0^{(1)}, f_1^{(0)}) + J_{2-1}(f_2^{(1)}, f_{-1}^{(0)}), \qquad f_s = f_{-s};$$

$$Lf_2^{(2)} = \frac{i}{2\alpha(U-c)}\Big\{\frac{\partial\Delta}{\mu\partial t}f_2^{(1)} - \nu\Delta^2 f_2^{(1)} - \varepsilon_1 J_2^{(2)}\Big\},$$

$$\Gamma f_2^{(2)} = 0,\ \psi_2^{(2)} = \frac{\partial}{\partial y}(f_2^{(1)} + \Phi_2^{(1)} + \mu w_2^{(2)}) = 0 \qquad (y = 0); \tag{23}$$

$$Lf_3^{(2)} = \frac{i\varepsilon_1}{3\alpha(U-c)}J_3^{(2)}, \qquad \Gamma f_3^{(2)} = 0, \qquad f_3^{(2)}(t,0) = 0. \tag{24}$$

The equation for $f_0^{(1)}$ follows from the condition $Q_0^{(2)} < \infty$ and can be written as

$$\Big(\frac{\partial}{\mu\partial t} - \nu\frac{\partial^2}{\partial y^2}\Big)\frac{\partial^2}{\partial y^2}f_0^{(1)} = \varepsilon_1 J_0^{(2)} \simeq \varepsilon_1\frac{d|A|^2}{dt}\frac{\partial}{\partial y}\Big(\frac{U''|\varphi_{11}|^2}{(U-c)^2}\Big), \tag{25}$$

with allowance for $Lf_1^{(1)} = Lw_1^{(1)} \ll Q_1^{(1)}$ $(y \gg 0)$. From (25), for $y \gg (t/\mathrm{Re}\,)^{1/2}$ we obtain the relation

$$\frac{\partial}{\partial y}f_0^{(1)} = \varepsilon_1\frac{\partial}{\partial t}|A|^2\frac{U''|\varphi_{11}|^2}{(U-c)^2} + \text{const},$$

which is asymptotic for $f_0^{(1)}$ as $y \to \infty$.

The correction $f_0^{(2)}$ is determined from similar reasoning for equations $\sim\mu^3$. The problem of determining ψ_0 for an isolated CL has been considered previously in [11, 12].

Thus, the solution outside the CL has the following form with accuracy to $O(\mu^3)$:

$$\begin{aligned}\tilde{\psi} = \Psi + \varepsilon A\varphi_{11}\,e^{i\theta_1} + \varepsilon\mu\{(f_1^{(1)} + w_1^{(1)} + B_1^{(1)}\varphi_{11} + D_1^{(1)}\varphi_{21})\,e^{i\theta_1} + f_2^{(1)}\,e^{2i\theta_1} + f_0^{(1)}\}\\ +\varepsilon\mu^2\{(f_1^{(2)} + w_1^{(2)} + B_1^{(2)}\varphi_{11} + D_1^{(2)}\varphi_{21})\,e^{i\theta_1} + (f_2^{(2)} + w_2^{(2)} + B_2^{(2)}\varphi_{12}\\ +D_2^{(2)}\varphi_{22})\,e^{2i\theta_1} + (f_0^{(2)} + w_0^{(2)} + \Phi_0^{(2)}) + f_3^{(2)}\,e^{3i\theta_1}\} + O(\varepsilon\mu^3),\end{aligned} \tag{26}$$

where

$$f_s^{(m)} = V_s^{-1}\Big(\varphi_{1s}\int_0^y \varphi_{2s}Q_s^{(m)}\,dy - \varphi_{2s}\int_0^y \varphi_{1s}Q_s^{(m)}\,dy\Big)$$

[see (10)], $w_s^{(m)}$, $B_s^{(m)}$, $D_s^{(m)}$, and $\Phi_0^{(m)}$ are determined by solving the internal equations (15) and (17). Formula (14) leads to the amplitude equation

$$\begin{aligned}\frac{d}{dt}A = \mu\Big\{\int_0^{y_\infty}\frac{U''|\varphi_{11}|^2dy}{i\alpha(U-c)}\Big\}^{-1}\Big\{\Delta D_1^{(1)} - w_1^{(1)}(t,0) + \mu\Big(\Delta D_1^{(2)} - w_1^{(2)}(t,0)\\ + \int_0^{y_\infty}\frac{i\varphi_{11}}{\alpha(U-c)}\Big(\frac{\partial\Delta}{\mu\partial t}\psi_1^{(1)} - \nu\Delta\frac{U''\varphi_{11}A}{U-c} - \varepsilon_1 J_1^{(2)}\Big)\,dy\Big) + O(\mu^2)\Big\}.\end{aligned} \tag{27}$$

The terms under the integration sign are due to the contribution of effects from regions outside the CL. To determine the internal solutions ψ_{iN} in the CL, we use a standard procedure of the method of matched asymptotic expansions. For this purpose, we expand $\tilde{\psi}$ into a series for $|z| = |y - y_1| \ll 1$, pass to the internal variables $\eta = z/\mu$, and collect terms of the same order with respect to the parameter μ.

The resulting relation is the boundary condition for (15):

$$\tilde{\psi} = \mu C\eta + \mu^2\frac{U_c'}{2}\eta^2 + \mu^3\frac{U_c''\eta^3}{6} + \ldots - \frac{\varepsilon A}{U_c'}e^{i\theta_1}\Big\{1 + \Big(\frac{U_c''}{U_c'}\eta - \sigma\Big)\mu\ln\Lambda$$
$$+\mu\Big(\frac{U_c''}{U_c'}\eta\ln|\eta| + \eta\Big(\frac{1}{y_1} - g_1\Big) + \frac{U_c'}{A}B_1^{(1)} - \sigma\ln|\eta| - \sigma - \frac{U_c'w_1^{(1)}}{A}\Big)$$
$$+\mu^2\ln\Lambda\Big(\frac{U_c''B_1^{(1)}}{A} - \sigma\Big(\frac{3}{2}\frac{U_c''}{U_c'} + g_1 - \frac{1}{y_1}\Big)\Big)\eta + \mu^2\Big(g_1'\eta^2 - \frac{U_c'}{A}D_1^{(1)}\eta - \frac{\sigma}{2}\eta$$
$$-B_1^{(1)}\frac{U_c''}{A}\eta\ln|\eta| + \frac{U_c'}{A}B_1^{(1)}\Big(\frac{1}{y_1} - g_1\Big) - \sigma\eta\ln|\eta|\Big(\frac{3}{2}\frac{U_c''}{U_c'} + g_1 - \frac{1}{y_1}\Big)\Big)\Big\} + O(\varepsilon^2\mu, \nu\mu). \tag{28}$$

Here $\Lambda = \mu/y_1$, $\sigma = (iU_c''/\alpha U_c'^3)(dA/Adt)$, $U_c = U(y_1)$, and $B_1^{(1)}$ and $D_1^{(1)}$ can take various values above and below the CL. Then, the solution ψ_{iN} for the internal equation (15) can be represented as

$$\psi_{iN} = \mu^2(\chi_1 + \mu\ln\mu\chi_2 + \mu\chi_3 + \mu^2\ln\mu\chi_4 + \ldots), \tag{29}$$

where $\chi_1 = U_c'\eta^2/2 - \varepsilon A\exp(i\theta_1)/(\mu^2U_c')$, $\chi_2 = \varepsilon A\exp(i\theta_1)/(\mu^2U_c')(\sigma - \eta U_c''/U_c')$, and χ_3 is found from the equation

$$L\chi_3 \equiv \Big\{\frac{\gamma}{\mu}\frac{\partial}{\partial\tau} + U_c'\eta\frac{\partial}{\partial\xi} + 2\frac{\varepsilon\alpha}{\mu^2U_c'}\,\mathrm{Im}\,(A\exp(i\alpha\xi))\frac{\partial}{\partial\eta} - \lambda\frac{\partial^2}{\partial\eta^2}\Big\}\frac{\partial^2}{\partial\eta^2}\chi_3 = 0 \tag{30}$$
$$\Big[\chi_3 = \frac{U_c''\eta^3}{6} - \frac{\varepsilon A\exp(i\theta_1)}{U_c'\mu^2}\Big\{\ln|\eta|\Big(\frac{U_c''}{U_c'}\eta - \sigma\Big) + \eta\Big(\frac{1}{y_1} - g_1\Big)$$
$$+\frac{U_c'B_1^{(1)}}{A} - \sigma - \frac{w_1^{(1)}U_c'}{A}\Big\}\quad(|\eta| \to \infty), \quad \xi = x - ct\Big].$$

The efficiency of contributions due to unsteady, nonlinear, and viscous effects, which determines the CL type, is determined by the relative values of the corresponding terms (γ/μ, ε/μ^2, and λ).

Taking into account the structure of boundary conditions (30), we easily obtain $\Delta B_1^{(1)} = 0$. The value of

$$B_1^{(1)} = \mathrm{Im}\,B_1^{(1)} = -\frac{d}{dt}A\frac{1}{\alpha U_c'}\Big(\frac{U_c''}{U_c'} + \frac{1}{y_1} - g_1\Big) \tag{31}$$

can be found without solving the internal problem and by integrating Eq. (30) with respect to η. In the next order,

$$\chi_4 = \frac{\varepsilon}{\mu^2}\frac{U_c''A\exp(i\theta_1)}{U_c'}\eta\Big(\sigma\Big(\frac{3}{2}\frac{U_c''}{U_c'} + g_1 - \frac{1}{y_1}\Big) - \frac{B_1^{(1)}}{A}\Big). \tag{32}$$

A solution for χ_5 is found from

$$L\chi_5 = -iU_c'\eta\varepsilon A\alpha/\mu^2 \qquad \Big[\chi_5 = -\frac{\varepsilon A}{\mu^2U_c'}\exp(i\theta_1)\Big\{g_1'\eta^2 - \frac{U_c'}{A}D_1^{(1)}\eta + B_1^{(1)}\frac{U_c''}{A}\eta\ln|\eta|$$
$$+\frac{U_c'}{A}B_1^{(1)}\eta\Big(\frac{1}{y_1} - g_1\Big) - \frac{\sigma\eta}{2} - \sigma\eta\ln|\eta|\Big(\frac{3}{2}\frac{U_c''}{U_c'} + g_1 - \frac{1}{y_1}\Big)\Big\} + \frac{U_c'''\eta^3}{6}\Big]. \tag{33}$$

Using (28), we can express $\Delta D_1^{(1)} \neq 0$ in terms of the integral of CL vorticity:

$$\Big\langle\frac{\partial}{\partial\eta}\tilde{\psi}(t,x,\infty) - \frac{\partial}{\partial\eta}\tilde{\psi}(t,x,-\infty)\Big\rangle = \varepsilon\mu^2\Delta D_1^{(1)} = \Big\langle\int_{-\infty}^{+\infty}\Omega d\eta\Big\rangle, \tag{34}$$

$$\langle\ldots\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi}\ldots d\theta, \qquad \Omega = \frac{\partial^2}{\partial\eta^2}\psi_{iN}.$$

Formula (34) is valid if $w^{(1)}$ does not contribute to the jump value $\partial\tilde{\psi}/\partial\eta$ in the CL. The latter follows explicitly from Eq. (17), which, for $y \to y_1$, is automatically transformed to (30): $L\partial^2 w^{(1)}/\partial\eta^2 = 0$. Note that for $\partial^2 w^{(1)}/\partial\eta^2$ as $y \to 0$ the WL characteristic scale changes ($U_c'\eta \to -U_c' y_1/\mu$), and inertial and viscous terms in (17) become comparable: $U_c' y_1 \partial^2 w^{(1)}/\partial y^2 \sim \lambda\mu\partial^4 w^{(1)}/\partial y^4$ for $y \sim \mathrm{Re}^{-1/2}$. Substantial nonlinearity is observed only if $\varepsilon A \sim 1$:

$$\varphi_{11} \sim y, \quad U_c' y_1 \frac{\partial^2 w^{(1)}}{\partial y^2} \sim \varepsilon A y \frac{\partial^3 w^{(1)}}{\partial y^3}.$$

Maximum contribution of this effect to the $w^{(1)}$ distribution is observed within the CL and decreases at its periphery $\sim\eta^{-1/2}$. The influence of the region $y \sim y_1$ on $w_1^{(1)}$ can be considerable even for $\mu \ll y_1$ [14]. In the case $y_1 \gg \mu \gg \mathrm{Re}^{-1/2}$, the CL effect on the formation of $w^{(1)}$ can be ignored, and Eq. (17) for $\eta > -\Lambda$ takes the form

$$\frac{\partial^2 w_1^{(1)}}{\partial\eta^2} \sim \eta^{-1/4}\exp\Big((i+1)\int\limits_0^{\eta}(\eta\ \mathrm{Re})^{1/2}\,d\eta\Big). \tag{35}$$

Under restrictions (34) and (35), the increment equation (27) coincides with that obtained in [8, 10]. It is easy to show that the proposed method is related to the "flexible wall" model [8, 9, 14]. Designating $F = A\varphi_{11} + \mu(f_1^{(1)} + \Phi_1^{(1)})$ and taking into account that $\Delta w_1^{(1)} \approx \partial^2 w_1^{(1)}/\partial y^2 + O(\alpha^2)$ from the boundary conditions for $y = 0$

$$F + \mu w_1^{(1)} = 0, \quad \frac{\partial}{\partial y}F + \mu\frac{\partial}{\partial y}w_1^{(1)} = O(\mu) \qquad \Big(\frac{\partial}{\partial y}w_1^{(1)} \sim \mu^{-1}\Big),$$

we obtain

$$\frac{1}{F}\frac{\partial F}{\partial y} \simeq \frac{\partial w_1^{(1)}}{w_1^{(1)}\partial y} = \int\limits_0^{\infty}\Delta w_1^{(1)}\,dy \Big/ \int\limits_0^{\infty}\int\limits_0^{\infty}\Delta w_1^{(1)}\,dy\,dy. \tag{36}$$

Equation (36) coincides with that used in [9, 14] up to designations. The accuracy of this model can be estimated using the method proposed.

Until now we considered an isolated CL ($\Lambda \ll 1$). For $\Lambda \sim 1$, the flow region near the wall ($y \leqslant y_1$) is not external. Solutions there should be constructed within the framework of internal equations (15) with the boundary conditions $\psi_{iN} = \partial\psi_{iN}/\partial\eta = 0$ ($\eta = -\Lambda^{-1}$) and (28) ($\eta \to \infty$). With allowance for $\Lambda \sim 1$, expansion (28) acquires the form of a power series (the terms $\sim\mu^n \ln\mu$ drop out):

$$\tilde{\psi} = \mu C\eta + \mu^2\Big\{\frac{U_c'}{2}\eta^2 - \frac{\varepsilon A\exp(i\theta_1)}{\mu^2 U_c'}(1+\Lambda\eta)\Big\} + \mu^3\Big\{\frac{U_c''\eta^3}{6} - \frac{\varepsilon A\exp(i\theta_1)}{\mu^2 U_c'}\Big[\Big(\frac{U_c''}{U_c'}\eta - \sigma\Big)\ln\Lambda + \frac{U_c''}{U_c'}\eta\ln|\eta|$$
$$-g_1\eta + (1+\Lambda\eta)\frac{U_c'}{A}B_1^{(1)} - \sigma(1+\Lambda\eta)\ln|\eta| - \sigma - \frac{U_c'}{A}w_1^{(1)} + \sigma\eta\Lambda\ln|\Lambda|\Big]\Big\} + O(\mu^4) = \sum_{n=1}\mu^n\tilde{\psi}_n, \tag{37}$$

where $w_1^{(1)}$ is an exponentially decaying function. A solution of the internal problem (15) is sought in the form

$$\psi_{iN} = \mu^2(\chi_1 + \mu\chi_2 + \mu^2\chi_3 + \ldots). \tag{38}$$

The role of solvability condition belongs to the first equality of (14), which establishes a relationship between the operators $P^{(m)}$ and the functions at the outer edge of the boundary layer.

From (37) and (38) we find

$$\chi_1 = \frac{U_c'}{2}\eta^2 - \frac{\varepsilon A\exp(i\theta_1)}{\mu^2 U_c'}(1+\Lambda\eta), \qquad F^{(1)} = -\frac{\varepsilon A\exp(i\theta_1)}{\mu^2 U_c'}(1+\Lambda\eta). \tag{39}$$

Then,

$$L_1\chi_2 \equiv \left\{\frac{\gamma}{\mu}\frac{\partial}{\partial\tau} + U_c'\eta\frac{\partial}{\partial\xi} + 2\frac{\varepsilon\alpha}{\mu^2 U_c'}\,\mathrm{Im}\,(A\exp(i\alpha\xi))(1+\Lambda\eta)\frac{\partial}{\partial\eta} - \lambda\frac{\partial^2}{\partial\eta^2}\right\}\frac{\partial^2\chi_2}{\partial\eta^2} = 0,$$
$$\chi_2 = \frac{U_c''\eta^3}{6} + w^{(2)} + F^{(2)}, \tag{40}$$

where $F^{(2)} + w^{(2)} = \partial F^{(1)}/\partial\eta + \mu\partial w^{(2)}/\partial\eta = 0$ $(\eta = -\Lambda)$, $\chi_2 \to \tilde{\psi}_3$ $(\eta\to\infty)$, and $\tilde{\psi}_3$ are the terms at μ^3 in (38).

In the next order with respect to μ, we have

$$L_1\chi_3 = 0, \quad \chi_3 = F^{(3)} + w^{(3)} + \frac{U_c'''\eta^4}{24}. \tag{41}$$

Here $F^{(3)} + w^{(3)} = \partial F^{(2)}/\partial\eta + \mu\partial w^{(3)}/\partial\eta = 0$ $(\eta = -\Lambda)$ and $\chi_3 \to \tilde{\psi}_4$ $(\eta\to\infty)$.

In constructing solutions for $\Lambda \gg 1$ and $\Lambda \sim 1$ we use the same principle, which is based on the difference in growth rate of the functions $F^{(m)}$ and $w^{(m)}$ in the critical layer. For $y_1 \sim \mu$, as noted above, it is not possible to split $F^{(m)}$ into $f^{(m)}$ and $\Phi^{(m)}$ in this region. It should be emphasized that in the linear approximation (the case of an l_ν layer) the solution obtained from (37)–(41) corresponds to that obtained in [15] by direct expansion. Artificial elimination of the terms $\sim\mu^n\ln\mu$, necessary for the analysis in [15], is performed automatically in this procedure.

Parametric Amplification of Subharmonics. The CL type determines the evolution of wave disturbances and, because of the differences in the transverse structure of the vortex field, affects the interaction efficiency of various spectrum components. The determining mechanism for the LTT at small initial intensities of fluctuations (S-transition) is the parametric excitation of low-frequency background pulsations. The resonance in the triad that includes a primary two-dimensional wave and a pair of spatially symmetric subharmonic waves determines the dominating mechanism of the S-transition [2–5]. It is of interest to examine how the formation of a nonlinear CL affects the resonance intensity. To clarify this issue, we consider the problem of parametric interaction of subharmonic waves in a flow with the $\tilde{\psi}$ determined by Eqs. (1) and (18). The averaging method will be used [2–5]. The field of velocities is sought in the form

$$\mathbf{v} = \left(\frac{\partial}{\partial y}\tilde{\psi}, -\frac{\partial}{\partial x}\tilde{\psi}, 0\right) + \varepsilon_2(v_1, v_2, v_3)b(t)\exp(i\theta_L)\cos\beta z,$$
$$\frac{db}{dt} = \gamma_L b + \varepsilon S_L b^*, \qquad \theta_L = \alpha_L x - \omega_L t, \tag{42}$$

where $\{v_i\}$ satisfy the three-dimensional Orr–Sommerfeld system with accuracy to $O(\varepsilon_2,\varepsilon)$ $(\varepsilon \gg \varepsilon_2)$; $\omega_L + i\gamma_L = \omega(\alpha_L, \pm\beta)$.

The coefficient $|S| = |S_L/A|$ characterizes the intensity of parametric coupling near the resonance $\omega_1 = 2\omega_L$ and $\alpha_1 \approx 2\alpha_L$. It is expressed in terms of solutions ψ_1 and $\{v_i\}$ [2] calculated with allowance for the mean flow profile ψ_0 (18). Figure 1 shows numerical values of $|S|$ for various Re in the case of substantially nonlinear $[(\lambda, \gamma/\mu) \ll \varepsilon A/\mu^2]$ and quasi-linear $(\lambda \geqslant \gamma/\mu \gg \varepsilon A/\mu^2)$ CLs which are asymptotically far from the wall $(y_1 \gg \mu)$. The nonlinear CL structure corresponds to that considered in [7] for steady nonlinear waves (A = const). Comparison with [7] for equilibrium values of A was used in computations as a test. The curve for a nonlinear primary wave (see Fig. 1, curve 2) is valid for the parameters α_1 and Re in the vicinity of the upper branch of the neutral curve of linear stability theory. The dashed part of curve 2 is a formal approximation in the region where $\Lambda \sim 1$.

The computation results indicate a considerable decrease in the coupling coefficients and, hence, the rates of parametric excitation of background subharmonics with attainment of a nonlinear CL regime. Taking into account the intense generation of higher harmonics and distortion of the mean flow that occur in the nonlinear regime, the result obtained supports the hypothesis [2, 5] on the correlation between the LTT scenario in the boundary layer and the CL regime. Indeed, according to the estimates of weakly nonlinear theory, typical values $\alpha \sim 10^{-1}$, Re $\sim 10^4$, and $|S| \sim 10$ yield a threshold amplitude $\tilde{A}_x \sim 10^{-3}$, for which

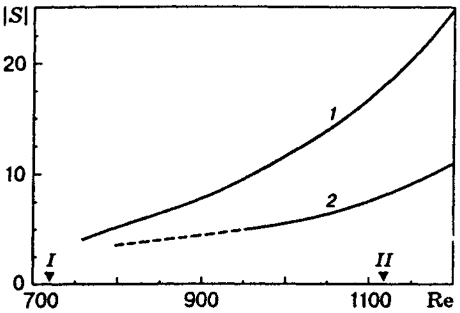

Fig. 1. Coefficients of parametric coupling $|S|$ versus Re for linear (1) and nonlinear (2) primary waves: $\omega_1/\mathrm{Re} = 115 \cdot 10^{-6}$, $\beta/\alpha_L \approx 2$, and I and II mark locations of the branches of the neutral stability curve for the primary waves.

the contributions of viscous and unsteady effects become comparable: $l_\nu \simeq (\alpha\mathrm{Re})^{-1/3} \sim l_t \simeq A_x|S|/\alpha$. In this case, $l_N \sim 10^{-3/2} < l_t$, and the nonlinear CL regime can occur only for $\varepsilon A > A_x$ in later stages of evolution. Specification of the initial amplitude close to the nonlinear CL threshold ($\tilde{A}_N \sim 10^{-2}$) can qualitatively modify the subsequent LTT pattern because of a decrease in the rate of resonant growth of low-frequency pulsations, which increases the time of attainment of an explosive regime. According to the experiments in [3], excess of the threshold value $\tilde{A}_N \sim 10^{-2}$ of the initial amplitude $\tilde{A}$ actually corresponds to the K-transition conditions, which fit the obtained estimate.

Finally, let us formulate the main conclusions.

• A new procedure for solving the problem of evolution of wave disturbances in boundary-layer flows was developed. This technique is suitable for both linear and weakly nonlinear evolution and also for intense (beyond the reach of weakly nonlinear theory) pulsations.

• The approach extends the results of the theory of disturbance evolution to the case of an arbitrary CL type and location with respect to the rigid wall. Wave interaction is taken into account.

• Coupling coefficients that determine the rate of parametric growth of subharmonic three-dimensional pulsations are computed. It is found that the growth rate of these pulsations decreases when they interact with a nonlinear primary wave, in contrast to a linear wave.

• From this result and analytical estimates of threshold transformation parameters of CL regimes, it is concluded that there is a direct relationship between the latter and LTT scenarios in the boundary layer. In this respect, the S-transition corresponds to a weakly nonlinear regime, whereas the K-transition is caused by the formation of a nonlinear CL of the primary wave.

• The above procedure of constructing evolution equations for two-dimensional, spatially periodic disturbances can be extended to the case of spatial–temporal evolution of three-dimensional waves. The main constraint of the method is connected with the requirement of fixed CL location during evolution.

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